(1) We have shown in various examples the existence of multiple poles in the scattering amplitude, the possibility of which has frequently been ignored.

(2) It has been suggested that the singularities of $\alpha_n(\nu)$ which occur off the positive real axis might be absent in a true field theory because of their connection with the fall into the center.[6] However, the scalar coupling theory considered here has displayed such singularities in spite of the fact that it has no possibility of collapse for physical l. The occurrence of these additional branch cuts in a complete theory cannot be excluded, and it would be almost remarkable if the consideration of recoil could completely eliminate them.

(3) We have noted that the trajectories associated with the models considered in this paper display marked differences in their qualitative behavior and analytic properties. All of these display analytic properties in conflict with those which have been expected to occur in a real field theory. It might well be anticipated, therefore, that the problem of analytic continuation in the complex angular momentum plane is not independent of the nature of the coupling.

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Artificial Singularity in the N/D Equations of the New Strip Approximation*

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The singularity introduced artificially into the equations of the new strip approximation, in order to bridge the gap between low and high energies, is investigated in detail. By explicit construction, it is shown that a necessary and sufficient condition for a (unique) solution of the N/D equations to exist is that the unitarity constraint on the cross section just above the strip boundary should be obeyed. The only singularities of the solution in the right-half angular momentum plane ($\mathrm{Re}J\geq 0$) are Regge poles.

I. INTRODUCTION

A SET of approximate dynamical equations based on the strip concept has recently been proposed for determining the self-consistent strong-interaction S matrix with Regge asymptotic behavior.[1] This paper is concerned with the singularity at the strip boundary introduced as a consequence of the approximation procedure. We propose to show that in spite of its artificial character this singularity plays a useful physical role and does not prevent a numerical solution of the equations. It also does not affect analyticity properties in angular momentum. The reader is assumed to be familiar with reference 1, whose notation is maintained here.

The integral equation in question is (III.11) of reference 1:

$$N_l(s)=B_l^P(s)+\frac{1}{\pi}\int_{s_0}^{s_1}ds'\,\frac{B_l^P(s')-B_l^P(s)}{s'-s}\rho_l(s')N_l(s'). \quad \text{(I.1)}$$

The singularity arises in the kernel because at $s=s_1$ the function $B_l^P(s)$ has a logarithmic branch point:

$$B_l^P(s)\underset{s\to s_1}{\longrightarrow}\frac{1}{\pi}\,\mathrm{Im}B_l^P(s_1)\ln(s_1-s). \quad \text{(I.2)}$$

Let us split off the singular part of the integral in (I.1):

$$N_l(s)=B_l^P(s)+\int_{s_0}^{s_1}ds'\,K_l(s,s')N_l(s')-\frac{\lambda_l}{\pi^2}\int_{s_0}^{s_1}ds'\,k(s,s')N_l(s'), \quad \text{(I.3)}$$

where

$$k(s,s')=\frac{\ln(s_1-s')-\ln(s_1-s)}{s'-s}, \quad \text{(I.4)}$$

$$\lambda_l=\rho_l(s_1)\,\mathrm{Im}B_l^P(s_1), \quad \text{(I.5)}$$

and where $K_l(s,s')$ is the residual part of the kernel obtained by comparison of Eqs. (I.1) and (I.3). In the dangerous region, $s\to s_1$, $s'\to s_1$,

$$K_l(s,s')\propto\frac{(s_1-s')\ln(s_1-s')-(s_1-s)\ln(s_1-s)}{s'-s}, \quad \text{(I.6)}$$

a behavior that causes no trouble. Equation (I.3) may

* Work done under auspices of U. S. Atomic Energy Commission.
[1] G. F. Chew, Phys. Rev. **129**, 2363 (1963).

be written as two coupled equations,

$$N_l^0(s)=B_l^P(s)+\int_{s_0}^{s_1} ds'\, K_l(s,s')N_l(s'), \qquad (I.7)$$

$$N_l(s)=N_l^0(s)-\frac{\lambda_l}{\pi^2}\int_{s_0}^{s_1} ds'\, k(s,s')N_l(s'), \qquad (I.8)$$

and we shall show in Sec. II that (I.8) can be explicitly solved to give

$$N_l(s)=\int_{s_0}^{s_1} ds'\, O_l(s,s')N_l^0(s'), \qquad (I.9)$$

where $O_l(s,s')$ is a known operator depending only on λ_l. Equation (I.7) then becomes a linear integral equation for $N_l^0(s)$:

$$N_l^0(s)=B_l^P(s)+\int_{s_0}^{s_1} ds'\, K_l'(s,s')N_l^0(s'), \qquad (I.10)$$

with

$$K_l'(s,s')=\int_{s_0}^{s_1} ds''\, K_l(s,s'')O_l(s'',s'). \qquad (I.11)$$

Finally, it will be shown that (I.10) is a nonsingular Fredholm equation.

II. THE OPERATOR $O_l(s,s')$

Directing our attention first to Eq. (I.8), we make the change of variables

$$x=\ln[(s_1-s_0)/(s_1-s)], \qquad (II.1)$$

which leads to

$$n_l(x)=n_l^0(x)+\frac{\lambda_l}{\pi^2}\int_0^\infty dx'\,\frac{x'-x}{e^{x'-x}-1}n_l(x'), \qquad (II.2)$$

if $n_l(x)=N_l(s(x))$, with a corresponding definition of $n_l^0(x)$. Now we have achieved the Wiener-Hopf form and may use the standard approach through Fourier or Laplace transforms.[2]

The key to the analysis is the asymptotic behavior as $x\rightarrow\infty$ (or as $s\rightarrow s_1$). It is immediately evident that there are no solutions of Eq. (II.2) diverging more strongly than e^x, but the physical requirements are sharper because, as we now show, a behavior $\propto e^{a_l x}$ implies that the limit of the phase shift as $s\rightarrow s_1$ is πa_l. Furthermore, the value of a_l is related to λ_l. To understand these points it is necessary to recall that in reference 1 the partial-wave amplitude was given by

$$B_l(s)=N_l(s)/D_l(s), \qquad (II.3)$$

where $D_l(s)$ is real analytic except for the cut between s_0 and s_1, along which it has the same phase as $B_l^{-1}(s)$, i.e., the phase $-\delta_l(s)$. $D_l(s)$ is normalized to unity at infinity and is finite except at $s=s_1$, so it must have the form

$$D_l(s)=\frac{(s-d_1)\cdots(s-d_{m_l})}{(s-s_1)^{m_l}}\times\exp\left[-\frac{1}{\pi}\int_{s_0}^{s_1} ds'\,\frac{\delta_l(s')-\delta_l(s_0)}{s'-s}\right], \qquad (II.4)$$

where $d_1\cdots d_{m_l}$ are the positions of the (real) zeros of $D_l(s)$ in the physical sheet. Then if $\delta_l(s)$ approaches a limit as $s\rightarrow s_1$, it follows that (to within logarithmic factors) one has

$$D_l(s)\underset{s\rightarrow s_1}{\propto}(s-s_1)^{-m_l}\exp\left\{-\frac{1}{\pi}[\delta_l(s_1)-\delta_l(s_0)]\ln(s-s_1)\right\}$$
$$=(s-s_1)^{-\{[\delta_l(s_1)-\delta_l(s_0)]/\pi+m_l\}}, \qquad (II.5)$$

or, if we adopt the convention that $\delta_l(s_0)=m_l\pi$,

$$D_l(s)\underset{s\rightarrow s_1}{\propto}(s-s_1)^{-\delta_l(s_1)/\pi}. \qquad (II.5')$$

Now, it was explained in reference 1 that elastic unitarity puts an upper bound on $B_l(s)$ for $s_0\leqslant s\leqslant s_1$ and that if $\mathrm{Im}B_l(s)$ approaches a limit as $s\rightarrow s_1$ this limit must be equal to $\mathrm{Im}B_l^P(s_1)\neq 0$. We, thus, deduce first that $N_l(s)$ should have the same limiting behavior as $D_l(s)$, i.e.,

$$N_l(s)\underset{s\rightarrow s_1}{\propto}(s-s_1)^{-\delta_l(s_1)/\pi}, \qquad (II.6)$$

and second that

$$\lim_{s\rightarrow s_1}\mathrm{Im}B_l(s)=\frac{\sin^2\delta_l(s_1)}{\rho_l(s_1)}=\mathrm{Im}B_l(s_1), \qquad (II.7)$$

or

$$\sin^2\delta_l(s_1)=\lambda_l.$$

It follows from formula (II.7) that λ_l must lie between 0 and 1 in the physically interesting case, and formula (II.6) gives the required physical interpretation of the asymptotic behavior as well as the connection with λ_l. Our final remarks before attacking Eq. (II.2) with Wiener-Hopf theory are that physically we expect both λ_l and $\delta_l(s_1)$ to approach zero as $l\rightarrow+\infty$, and that we hope we are dealing with a solution analytic in l. It is, therefore, the first-quadrant branch of formula (II.7) that is of interest, where $0<\delta_l<\pi/2$.

Let us now consider the Fourier transforms of the various terms in Eq. (II.2), defining

$$g_l^{(+)}(k)=\frac{1}{(2\pi)^{1/2}}\int_0^\infty dx\, e^{ikx}n_l(x), \qquad (II.8)$$

$$g_l^{(-)}(k)=\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^0 dx\, e^{ikx}n_l(x), \qquad (II.9)$$

$$g_l^0(k)=\frac{1}{(2\pi)^{1/2}}\int_0^\infty dx\, e^{ikx}n_l^0(x), \qquad (II.10)$$

[2] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company, Inc., New York, 1953), Part I, p. 990.

with the convention that $n_l^0(x)=0$ for $x<0$. If $n_l(x) \propto \exp(a_l x)$ as $x\to+\infty$, then $g_l^{(+)}(k)$ is well defined and holomorphic for $\mathrm{Im}k>a_l$, while from Eq. (II.2) we see that $g_l^{(-)}(k)$ is holomorphic for $\mathrm{Im}k<1$. An examination of the definition of $n_l^0(x)$ shows that $g_l^0(k)$ is holomorphic for $\mathrm{Im}k>0$ (remember that $a_l<1$). We may then take the Fourier transform of Eq. (II.2) anywhere in a strip such that $0<\mathrm{Im}k<1$, $\mathrm{Im}k>a_l$, to obtain

$$g_l^{(+)}(k)[1-\lambda_l R(k)]+g_l^{(-)}(k)=g_l^0(k), \quad \text{(II.11)}$$

where

$$R(k)=\frac{1}{\pi^2}\int_{-\infty}^{\infty}dx\, e^{-ikx}\frac{x}{e^x-1}, \\ =1/\sin^2(\pi ik). \quad \text{(II.12)}$$

The function $R(k)$ is holomorphic in the strip $0<\mathrm{Im}k<1$ and the function $1-\lambda_l R(k)$ is similarly holomorphic but with a pair of zeros when $\sin^2(\pi ik)=\lambda_l$, that is, at

$$k_{1l}=(i/\pi)\delta_l(s_1), \quad \text{(II.13)}$$

and

$$k_{2l}=(i/\pi)[\pi-\delta_l(s_1)]. \quad \text{(II.14)}$$

As $\lambda_l\to 0$ the positions of these zeros approach 0 and i, respectively, while as $\lambda_l\to 1$ they converge on the midpoint of the strip at $i/2$. Since we want a solution such that $a_l=\delta_l(s_1)/\pi$, it is permissible and desirable for $g_l^{(+)}(k)$ to have a pole at $k=k_{1l}$ but not at $k=k_{2l}$. The problem is to construct a function $g_l^{(+)}(k)$ consistent with Eq. (II.11) and with its uppermost singularity at $k=k_{1l}$.

To achieve this end we write

$$1-\frac{\lambda_l}{\sin^2(\pi ik)}=\frac{\phi_{2l}(k)}{\phi_{1l}(k)}, \quad \text{(II.15)}$$

where[3]

$$\phi_{1l}(k)=[\Gamma(-ik+a_l)\Gamma(-ik-a_l)]/\Gamma^2(-ik) \quad \text{(II.16)}$$

is holomorphic and free from zeros for $\mathrm{Im}k>\mathrm{Im}k_{1l}$, while

$$\phi_{2l}(k)=\Gamma^2(1+ik)/\Gamma(1+ik-a_l)\Gamma(1+ik+a_l) \quad \text{(II.17)}$$

is holomorphic and free from zeros for $\mathrm{Im}k<\mathrm{Im}k_{2l}$. Evidently, $\phi_{2l}(k)$ has a simple zero at $k=k_{2l}$, while $\phi_{1l}(k)$ has a simple pole at $k=k_{1l}$, the remaining zeros and poles of ϕ_{1l} and ϕ_{2l} lying outside the strip $0<\mathrm{Im}k<1$. Both $\phi_{1l}(k)$ and $\phi_{2l}(k)$ approach constants as $|k|\to\infty$ within the appropriate half-planes of analyticity. Let us divide Eq. (II.11) by $\phi_{2l}(k)$ to obtain

$$\frac{g_l^{(+)}(k)}{\phi_{1l}(k)}+\frac{g_l^{(-)}(k)}{\phi_{2l}(k)}=\frac{g_l^0(k)}{\phi_{2l}(k)}, \quad \text{(II.18)}$$

and then split the right-hand side into a piece analytic for $\mathrm{Im}k>0$ and a piece analytic for $\mathrm{Im}k<\mathrm{Im}k_{2l}$:

$$\frac{g_l^0(k)}{\phi_{2l}(k)}=\frac{1}{2\pi i}\int_{-\infty+i\epsilon}^{\infty+i\epsilon}\frac{dk'}{k'-k}\frac{g_l^0(k')}{\phi_{2l}(k')} \\ -\frac{1}{2\pi i}\int_{-\infty+k_{2l}-i\epsilon}^{\infty+k_{2l}-i\epsilon}\frac{dk'}{k'-k}\frac{g_l^0(k')}{\phi_{2l}(k')}. \quad \text{(II.19)}$$

The former we identify with the first term on the left-hand side of Eq. (II.18), and the latter with the second term [remembering that if $n_l(x)$ is finite except at ∞ then $g_l^{(+)}(k)$ and $g_l^{(-)}(k)$ separately vanish as $|k|\to\infty$ within the appropriate half-planes], giving

$$g_l^{(+)}(k)=\frac{\phi_{1l}(k)}{2\pi i}\int_{-\infty+i\epsilon}^{\infty+i\epsilon}\frac{dk'}{k'-k}\frac{g_l^0(k')}{\phi_{2l}(k')}, \quad \text{(II.20)}$$

$$g_l^{(-)}(k)=-\frac{\phi_{2l}(k)}{2\pi i}\int_{-\infty+k_{2l}-i\epsilon}^{\infty+k_{2l}-i\epsilon}\frac{dk'}{k'-k}\frac{g_l^0(k')}{\phi_{2l}(k')}. \quad \text{(II.21)}$$

Inspection of these formulas shows that our objective has been achieved. The function $g_l^{(+)}(k)$ is meromorphic for $\mathrm{Im}k>0$ with a simple pole at $k=k_{1l}$, while $g_l^{(-)}(k)$ is holomorphic for $\mathrm{Im}k<1$.

The operator $\theta_l(x,x')$, such that

$$n_l(x)\underset{x\geq 0}{=}\int_0^\infty dx'\,\theta_l(x,x')n_l^0(x'),$$

is given by

$$\theta_l(x,x')=\frac{1}{(2\pi)^2 i}\int_C dk\int_{C'}dk'\frac{e^{ik'x'-ikx}}{k'-k}\frac{\phi_{1l}(k)}{\phi_{2l}(k')}. \quad \text{(II.22)}$$

where the horizontal contour C passes above k_{1l} while C' passes below C and also below k_{2l}. The asymptotic behavior of $\theta_l(x,x')$ may then be inferred to be

$$\theta_l(x,x') \underset{\substack{x\to\infty \\ x'\text{ fixed}}}{\propto} e^{-ik_{1l}x}=e^{a_l x}, \\ \underset{\substack{x'\to\infty \\ x\text{ fixed}}}{\propto} e^{+ik_{2l}x}=e^{(a_l-1)x'}.$$

Changing variables back to s, s', one has

$$O_l(s,s')=\theta_l(x(s),x(s'))/(s_1-s'),$$

so that

$$O_l(s,s') \underset{\substack{s\to s_1 \\ s'\text{ fixed}}}{\propto} (s_1-s)^{-a_l}, \\ \underset{\substack{s'\to s_1 \\ s\text{ fixed}}}{\propto} (s_1-s')^{-a_l}. \quad \text{(II.23)}$$

III. THE FREDHOLM EQUATION

It remains to be established that Eq. (I.10) is of the Fredholm type, or specifically that

[3] I am indebted to J. R. Taylor for these expressions.

$$\int_{s_0}^{s_1}\int_{s_0}^{s_1} dsds'|K_l'(s,s')|^2<\infty.$$

The upper limit is the dangerous point, but as long as $a_l<\frac{1}{2}$ we see from Eq. (II.23) that there is no trouble. The Fredholm form has indeed been restored.

Perhaps the most immediate subsequent question is whether our new kernel $K_l'(s,s')$ is holomorphic in l over the same domain as $B_l^P(s)$. This is equivalent to the corresponding question about $O_l(s,s')$, which then leads us to an examination of (II.22). Evidently, as long as $0<\lambda_l<1$, so that $\mathrm{Im}k_{2l}>\mathrm{Im}k_{1l}$, we are dealing with an analytic function of l wherever λ_l as given by (I.5) is analytic. Now it will certainly happen that, for some choices of s_1 or some guesses about Regge trajectories and residues for the crossed channels, we shall find from the formulas of reference 1 that $\lambda_l>1$ or $\lambda_l<0$ for some $\mathrm{Re}l\gtrsim0$. When this catastrophe occurs, however, it is a sign either that we have made a bad guess or that the aforementioned formulas are insufficiently accurate, because in an exact calculation unitarity requires $0\leqslant\lambda_l\leqslant1$. Thus, if physically reasonable solutions of the strip equations can be found they will have the property that the only singularities in the right half l plane are Regge poles, arising from the zeros of $D_l(s)$. It is expected that $B_l^P(s)$ and, therefore, λ_l has fixed singularities in the left half l plane. By analogy with potential scattering one might expect Regge trajectories to terminate at these points, but the continuation based on our approximate equations must fail somewhat sooner, when λ_l exceeds the unitarity bounds.

It follows, incidentally, from the manner in which our N/D equations have been constructed that both $\mathrm{Re}B_l(s)$ and $\mathrm{Im}B_l(s)$ are continuous through the point s_1. In a one-channel approximation this means that the inelastic cross section vanishes at $s=s_1$ and rises gradually to the correct Regge limit. If a generalization of the equations in this paper to several two-body channels can be made, a more realistic inelastic threshold can be achieved.

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Approach to Equilibrium in Quantal Systems. II. Time-Dependent Temperatures and Magnetic Resonance

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The present paper contains an extension and generalization of results in a previous paper on the basis of the master equation for the approach to equilibrium of a system of interest. The concept of quasi-equilibrium of the system of interest associated with a time-dependent temperature is introduced and is then applied to a description of the processes of longitudinal and transverse relaxation in magnetic resonance and to a discussion of the law of entropy variation. Systems of interest of "size" comparable to their surroundings are consistently included in the treatment.

I. INTRODUCTION

IN a previous paper[1] the "master" or Boltzmann "gain-loss" equation was derived from the Schrödinger equation for an isolated "supersystem" [$A+B$] composed of a "system of interest" [A] in relatively weak interaction with a larger system called the "surroundings" [B]. The random phase assumption was required for the state of the supersystem [$A+B$] at the *initial* time only. The Hamiltonian $\mathcal{H}$ of such a supersystem is

$$\mathcal{H}=\mathcal{H}_{[A]}^{(0)}+\mathcal{H}_{[B]}^{(0)}+V, \tag{1}$$

where $\mathcal{H}_{[A]}^{(0)}$ contains only [A] system dynamical variables, $\mathcal{H}_{[B]}^{(0)}$ only [B] dynamical system variables, and V dynamical variables of both systems. A master equation for the occupation probabilities of the system of interest was then derived in I under the assumption that the surroundings [B] have a large internal energy com-

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[1] A. Sher and H. Primakoff, Phys. Rev. **119**, 178 (1960). This paper will be referred to as I in the present work.